Modelling Geoadditive Regression Data

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joint work with
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29.11.2007

Outline

- Geoadditive Regression: An Application to Car Insurance Data.
- Bayesian Inference in Structured Additive Regression.
- Spatio-Temporal Regression: Forest Health Data.

Structured Additive Regression

- Regression in a general sense:
  - Generalised linear models,
  - Multivariate (categorical) generalised linear models,
  - Regression models for duration times (Cox-type models, multi-state models).

- Common structure: Model a quantity of interest in terms of categorical and continuous covariates, e.g.

$$\mathbb{E}(y|u) = h(u'\gamma) \qquad \text{(GLM)}$$

or

$$\lambda(t|u) = \lambda_0(t)\exp(u'\gamma) \qquad \text{(Cox model)}$$

- General idea of structured additive regression: Replace usual parametric predictor with a flexible semiparametric predictor containing
  - Nonparametric effects of time scales and continuous covariates,
  - Spatial effects,
  - Interaction surfaces,
  - Varying coefficient terms (continuous and spatial effect modifiers),
  - Random intercepts and random slopes.

- Example: Car insurance data from two insurance companies in Belgium.

- Sample of approximately 160.000 policyholders.

- Aims: Separate risk analyses for claim size and claim frequency to predict risk premium from covariates.

- Variables of primary interest: Claim size y_i or claim frequency h_i of policyholders.

- Covariates:

$vage$	vehicles age
$page$	policyholders age
hp	vehicles horsepower
bm	bonus-malus score
s	district in Belgium
v	Vector of categorical covariates

- Geoadditive models:
  - Gaussian model for log-costs $\log(y)$:

$$\log(y) \sim N(\eta, \sigma^2)$$

with

$$\eta = f_1(vage) + f_2(page) + f_3(bm) + f_4(hp) + f_{spat}(s) + v'\zeta.$$

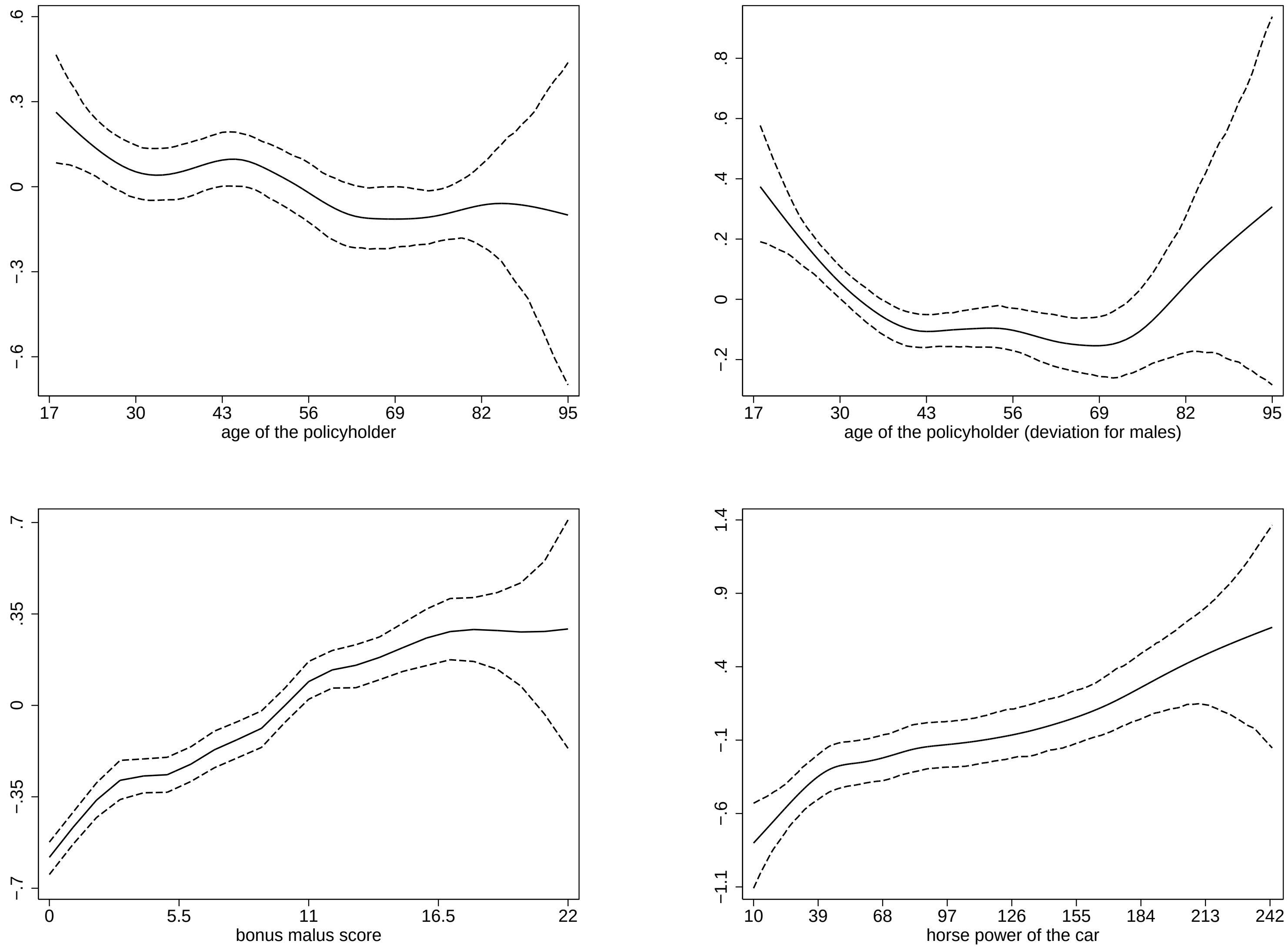
  - Poisson model for frequencies h_i:

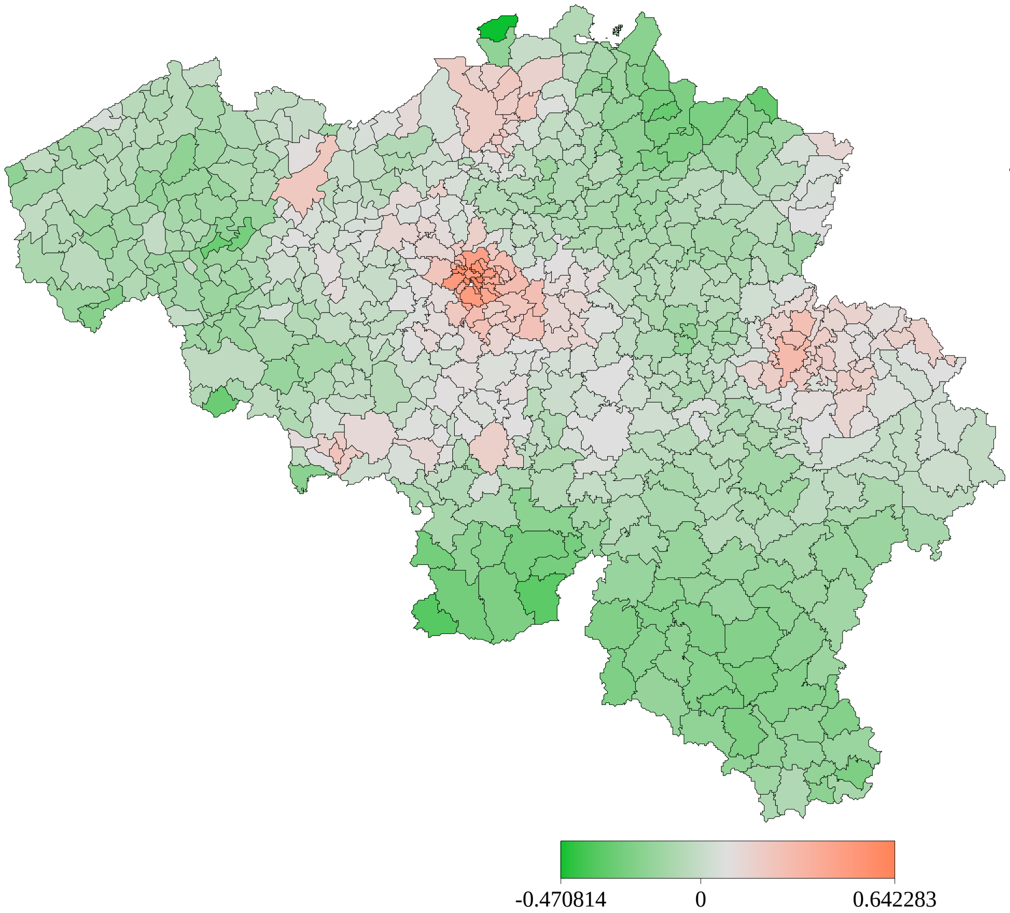
$$h \sim Po(\exp(\eta))$$

with

$$\eta = f_1(vage) + f_2(page) + f_3(page)sex + f_3(bm) + f_4(hp) + f_{spat}(s) + v'\zeta.$$

- Results for claim frequency:

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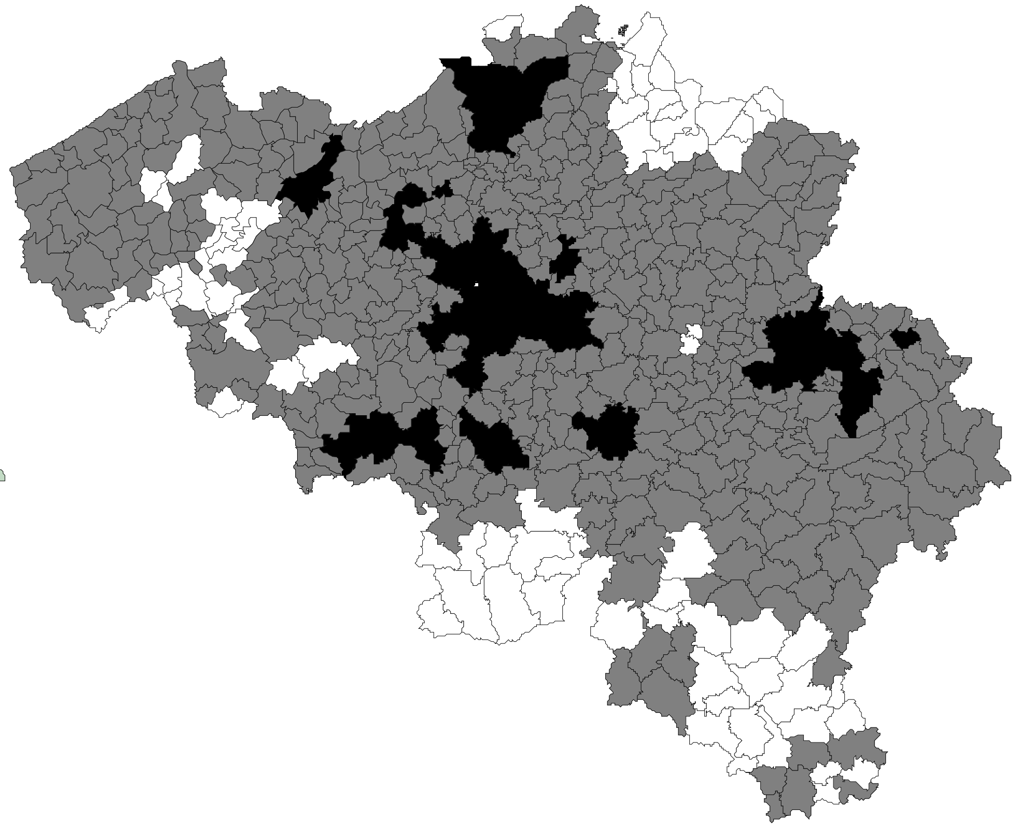
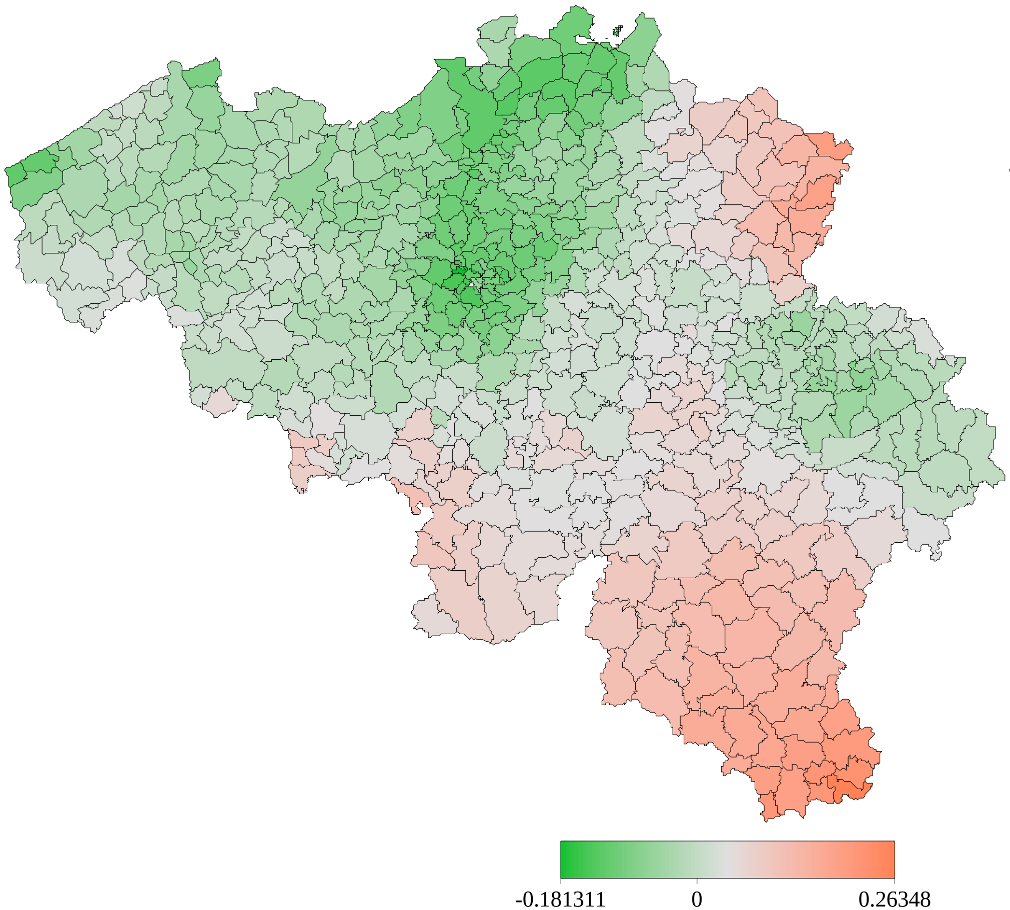
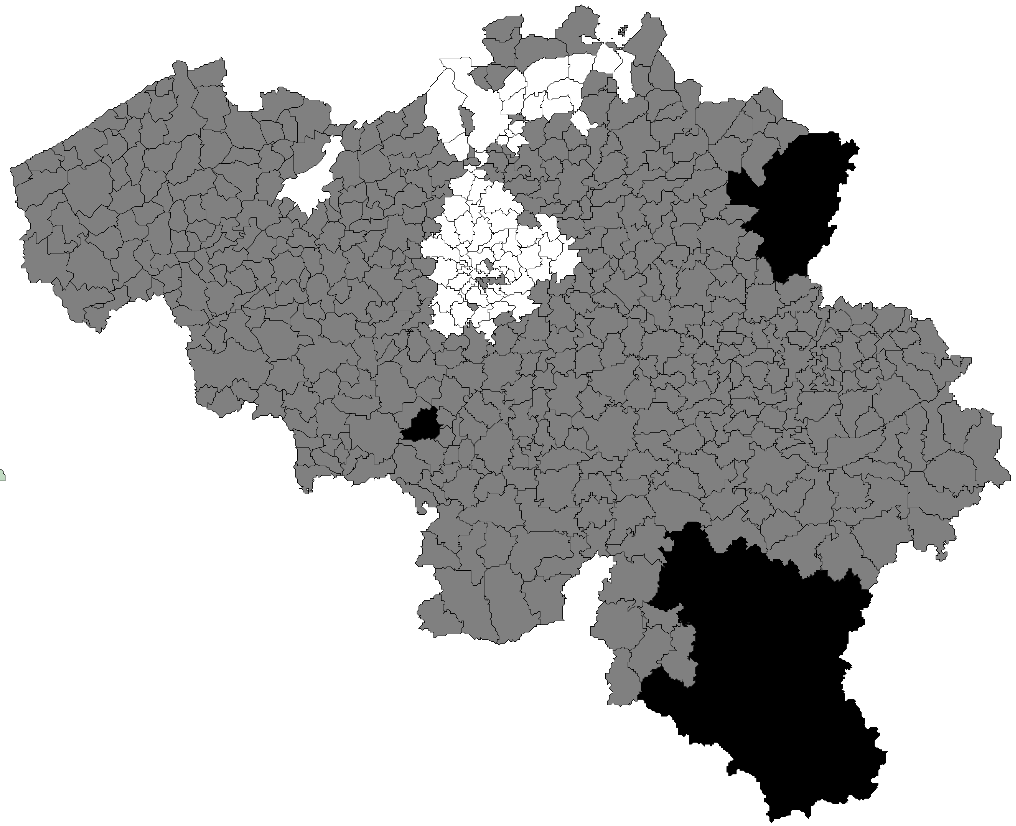


- Spatial effect for claim size:



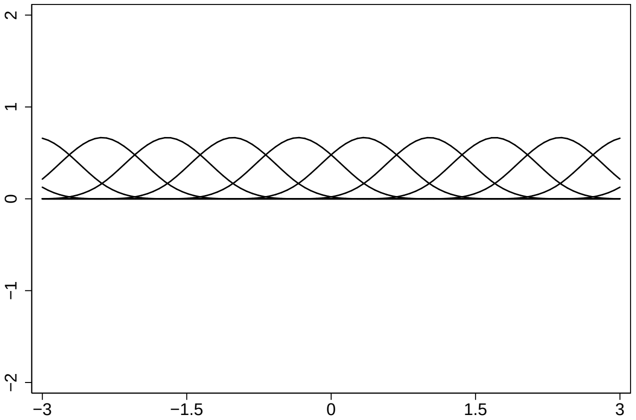


Model Components and Priors

- Penalised splines.
  - Approximate $f(x) = \sum \xi_j B_j(x)$ by a weighted sum of B-spline basis functions.
  - Employ a large number of basis functions to enable flexibility.
  - Penalise differences between parameters of adjacent basis functions to ensure smoothness

$$\frac{1}{2\tau^2}\sum(\xi_j - \xi_{j-1})^2 \qquad \text{(first order differences)}$$

$$\frac{1}{2\tau^2}\sum(\xi_j - 2\xi_{j-1} + \xi_{j-2})^2 \qquad \text{(second order differences)}$$



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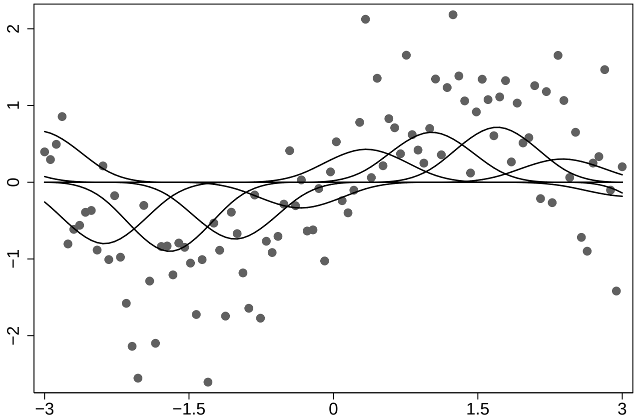

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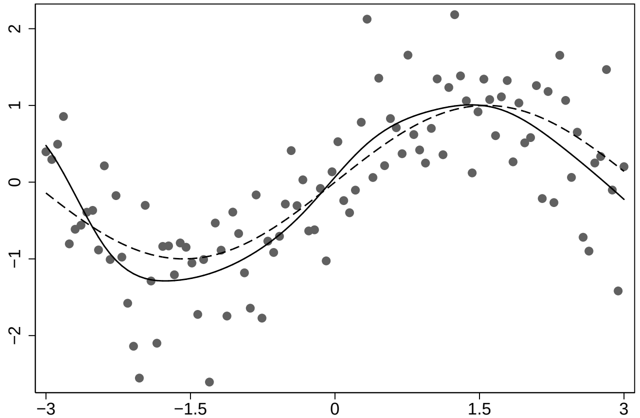

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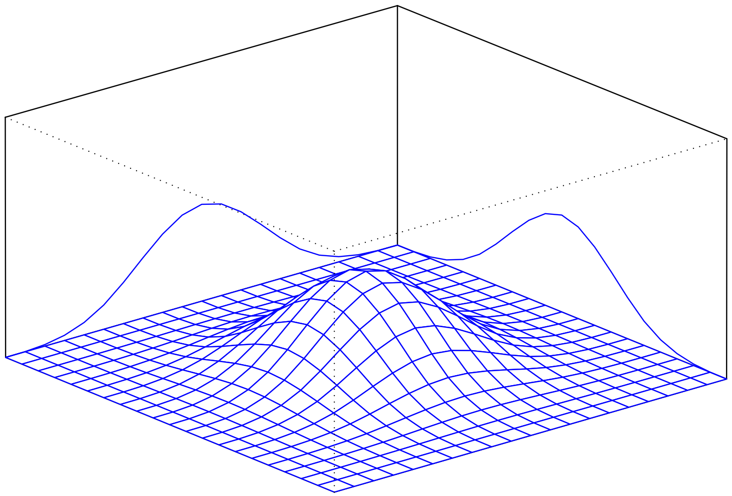
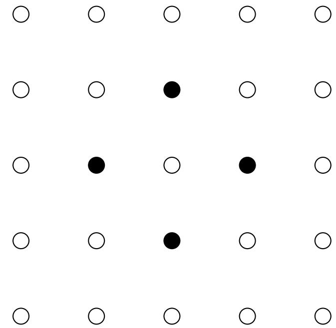
- Bivariate penalised splines.



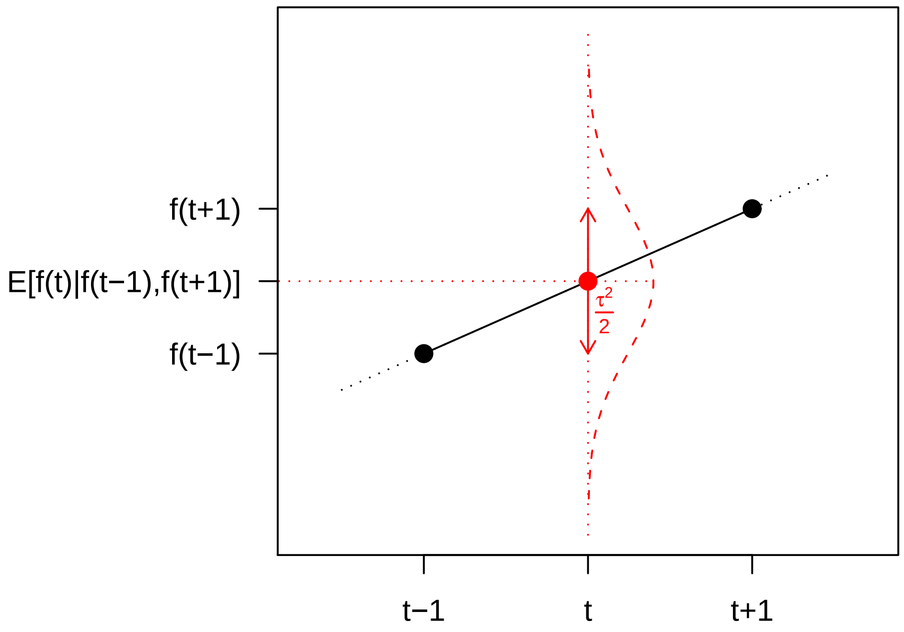


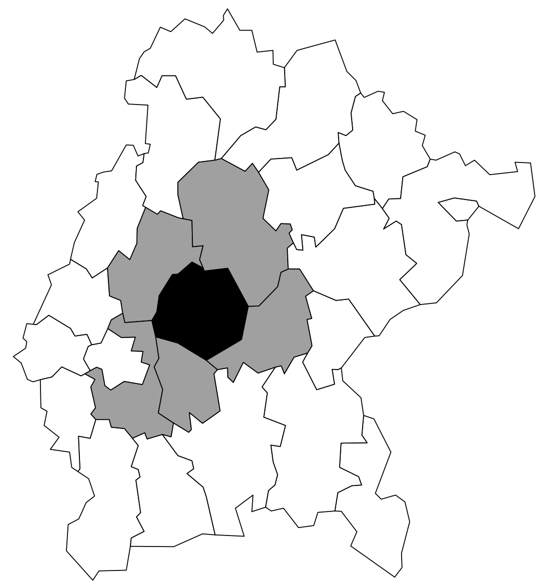
- Varying coefficient models.
  - Effect of covariate x varies smoothly over the domain of a second covariate z:

$$f(x,z) = x \cdot g(z)$$

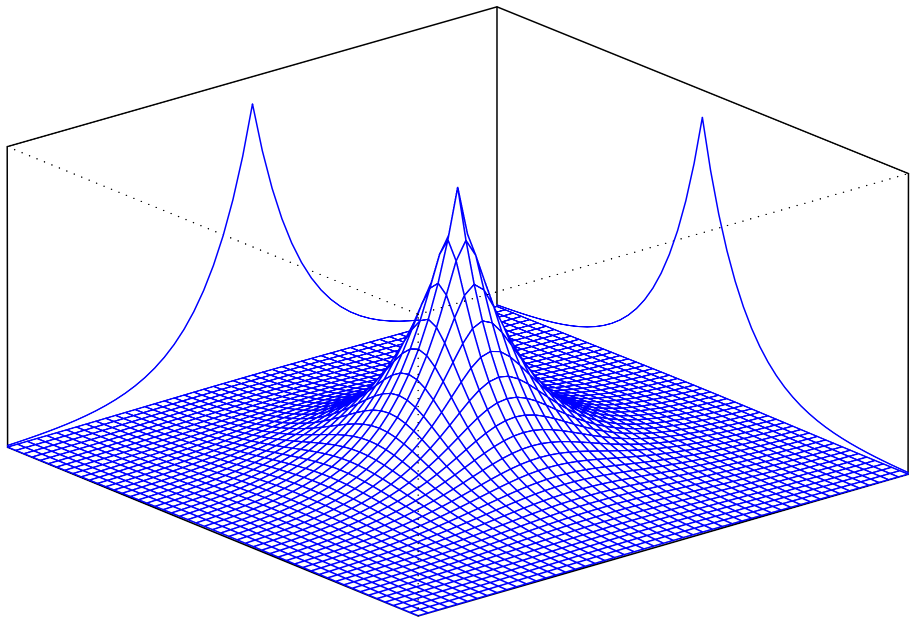
  - Spatial effect modifier $\Rightarrow$ Geographically weighted regression.

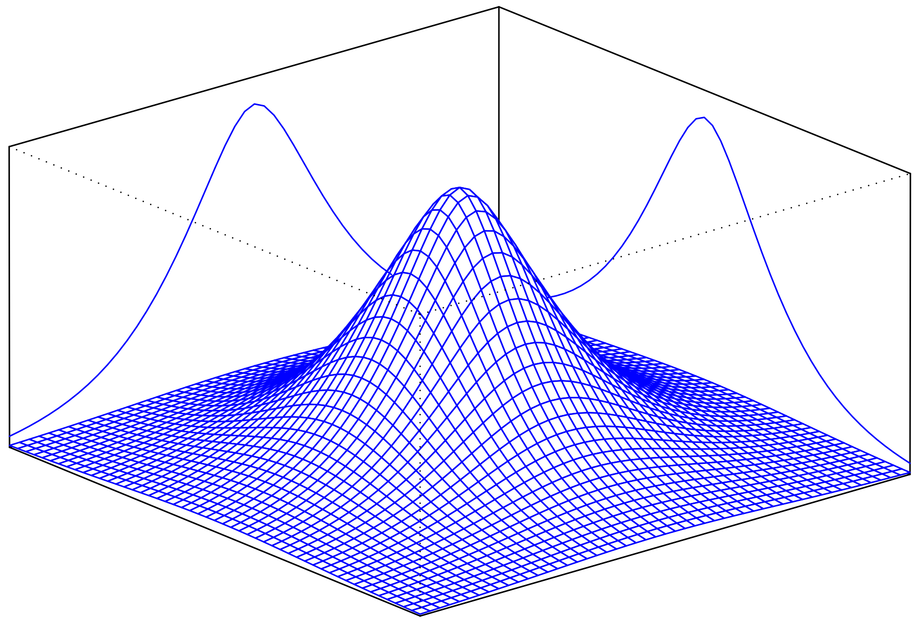
- Spatial effect for regional data: Markov random fields.
  - Bivariate extension of a first order random walk on the real line.
  - Define appropriate neighbourhoods for the regions.
  - Assume that the expected value of $f_{spat}(s)$ is the average of the function evaluations of adjacent sites.





- Spatial effect for point-referenced data: Stationary Gaussian random fields.
  - Well-known as Kriging in the geostatistics literature.
  - Spatial effect follows a zero mean stationary Gaussian stochastic process.
  - Correlation of two arbitrary sites is defined by an intrinsic correlation function.
  - Can be interpreted as a basis function approach with radial basis functions.





- All effects can be cast into one general framework.

- All vectors of function evaluations f_j can be expressed as

$$f_j = Z_j \xi_j$$

with design matrix Z_j and regression coefficients ξ_j.

- Generic form of the prior for ξ_j:

$$p(\xi_j|\tau_j^2) \propto (\tau_j^2)^{-\frac{k_j}{2}} \exp\left(-\frac{1}{2\tau_j^2}\xi_j' K_j \xi_j\right).$$

- $K_j \geq 0$ acts as a penalty matrix, $\mathrm{rank}(K_j) = k_j \leq d_j = \dim(\xi_j)$.

- $\tau_j^2 \geq 0$ can be interpreted as a variance or (inverse) smoothness parameter.

Bayesian Inference

- Fully Bayesian inference:
  - All parameters (including the variance parameters τ^2) are assigned suitable prior distributions.
  - Typically, estimation is based on MCMC simulation techniques.
  - Usual estimates: Posterior expectation, posterior median (easily obtained from the samples).
- Empirical Bayes inference:
  - Differentiate between parameters of primary interest (regression coefficients) and hyperparameters (variances).
  - Assign priors only to the former.
  - Estimate the hyperparameters by maximising their marginal posterior.
  - Plugging these estimates into the joint posterior and maximising with respect to the parameters of primary interest yields posterior mode estimates.

- MCMC-based inference:
  - Assign inverse gamma prior to τ_j^2:

$$p(\tau_j^2) \propto \frac{1}{(\tau_j^2)^{a_j+1}} \exp\left(-\frac{b_j}{\tau_j^2}\right).$$

Proper for	$a_j > 0,\ b_j > 0$	Common choice: $a_j = b_j = \varepsilon$ small.
Improper for	$b_j = 0,\ a_j = -1$	Flat prior for variance τ_j^2,
	$b_j = 0,\ a_j = -\frac{1}{2}$	Flat prior for standard deviation τ_j.

  - Conditions for proper posteriors in structured additive regression are available.
  - Gibbs sampler for $\tau_j^2|\cdot$:

    Sample from an inverse Gamma distribution with parameters

$$a_j' = a_j + \frac{1}{2}\text{rank}(K_j) \qquad \text{and} \qquad b_j' = b_j + \frac{1}{2}\xi_j' K_j \xi_j.$$

– Metropolis-Hastings update for $\xi_j|\cdot$:

Propose new state from a multivariate Gaussian distribution with precision matrix and mean

$$P_j = Z_j' W Z_j + \frac{1}{\tau_j^2} K_j \qquad \text{and} \qquad m_j = P_j^{-1} Z_j' W (\tilde{y} - \eta_{-j}).$$

IWLS-Proposal with appropriately defined working weights W and working observations $\tilde{y}$.

- Efficient algorithms make use of the sparse matrix structure of P_j and K_j.

- Empirical Bayes inference.
  - Consider the variances τ_j^2 as unknown constants to be estimated from their marginal posterior.
  - Consider the regression coefficients ξ_j as correlated random effects with multivariate Gaussian distribution

    $\Rightarrow$ Use mixed model methodology for estimation.

- Problem: In most cases partially improper random effects distribution.

- Mixed model representation: Decompose

$$\xi_j = X_j \beta_j + V_j b_j,$$

where

$$p(\beta_j) \propto const \qquad \text{and} \qquad b_j \sim N(0, \tau_j^2 I_{k_j}).$$

$\Rightarrow$ β_j is a fixed effect and b_j is an i.i.d. random effect.

- This yields a variance components model with pedictor

$$\eta = X\beta + Vb$$

where in turn

$$p(\beta) \propto const \qquad \text{and} \qquad b \sim N(0, Q).$$

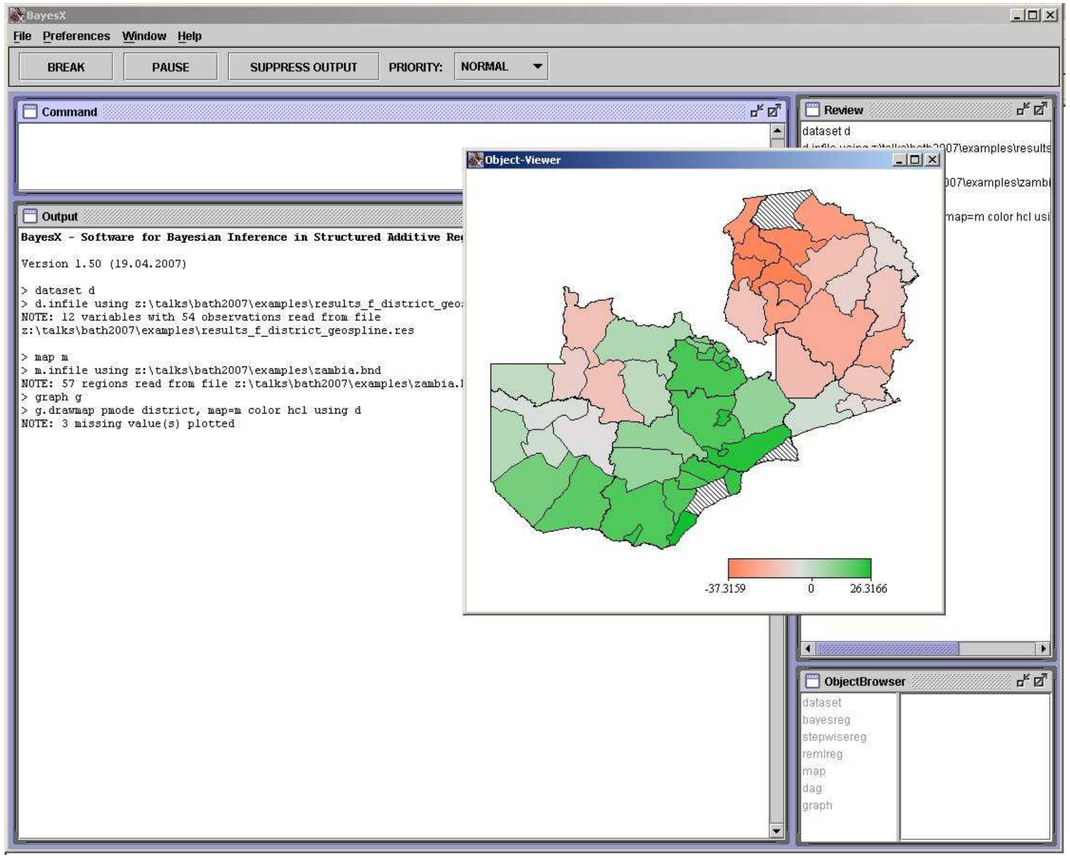
- Obtain empirical Bayes estimates / penalized likelihood estimates via iterating
  - Penalized maximum likelihood for the regression coefficients β and b.
  - Restricted Maximum / Marginal likelihood for the variance parameters in Q:

$$L(Q) = \int L(\beta, b, Q) p(b) d\beta db \rightarrow \max_{Q}.$$

- Involves a Laplace approximation to the marginal likelihood (corresponding to REML estimation of variances in Gaussian mixed models).

BayesX

- BayesX is a software tool for estimating structured additive regression models.



- Available from

`http://www.stat.uni-muenchen.de/~bayesx`

Spatio-Temporal Regression: Forest Health Data

- Yearly forest health inventories carried out from 1983 to 2004.

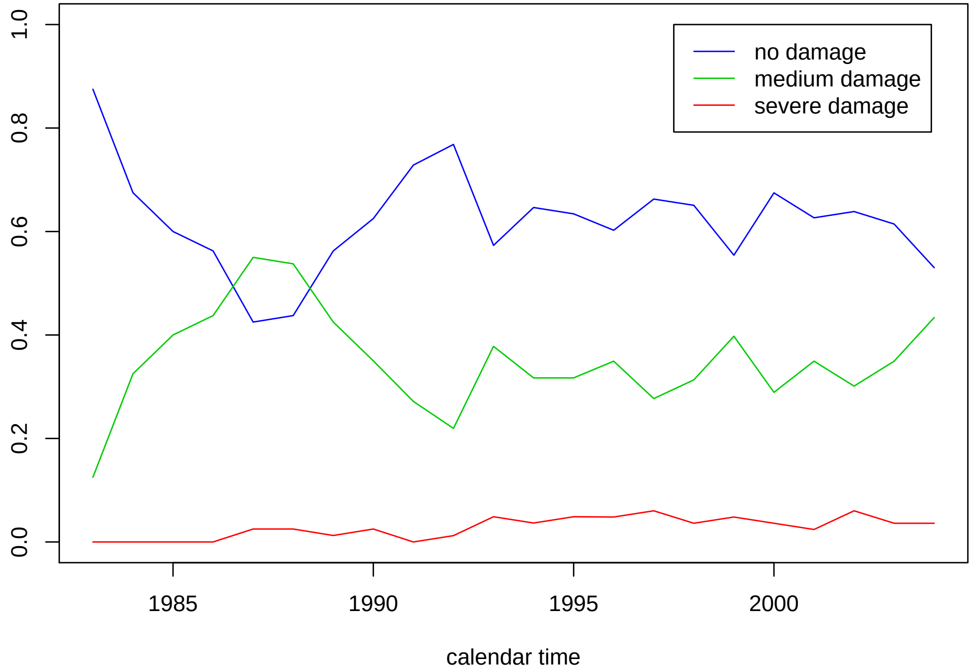
- 83 beeches within a 15 km times 10 km area.

- Response: defoliation degree of beech i in year t, measured in three ordered categories:

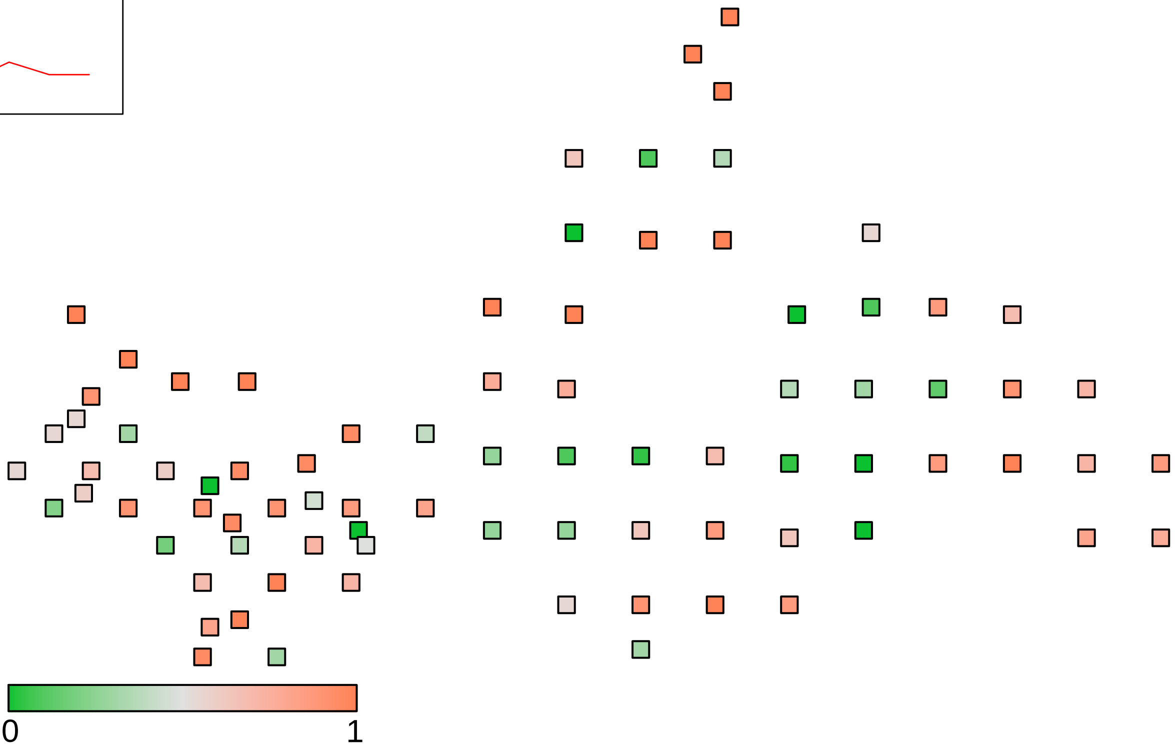
  $y_{it} = 1$ no defoliation,
  $y_{it} = 2$ defoliation 25% or less,
  $y_{it} = 3$ defoliation above 25%.

- Covariates:

  t calendar time,
  s_i site of the beech,
  a_{it} age of the tree in years,
  u_{it} further (mostly categorical) covariates.



Empirical time trends.



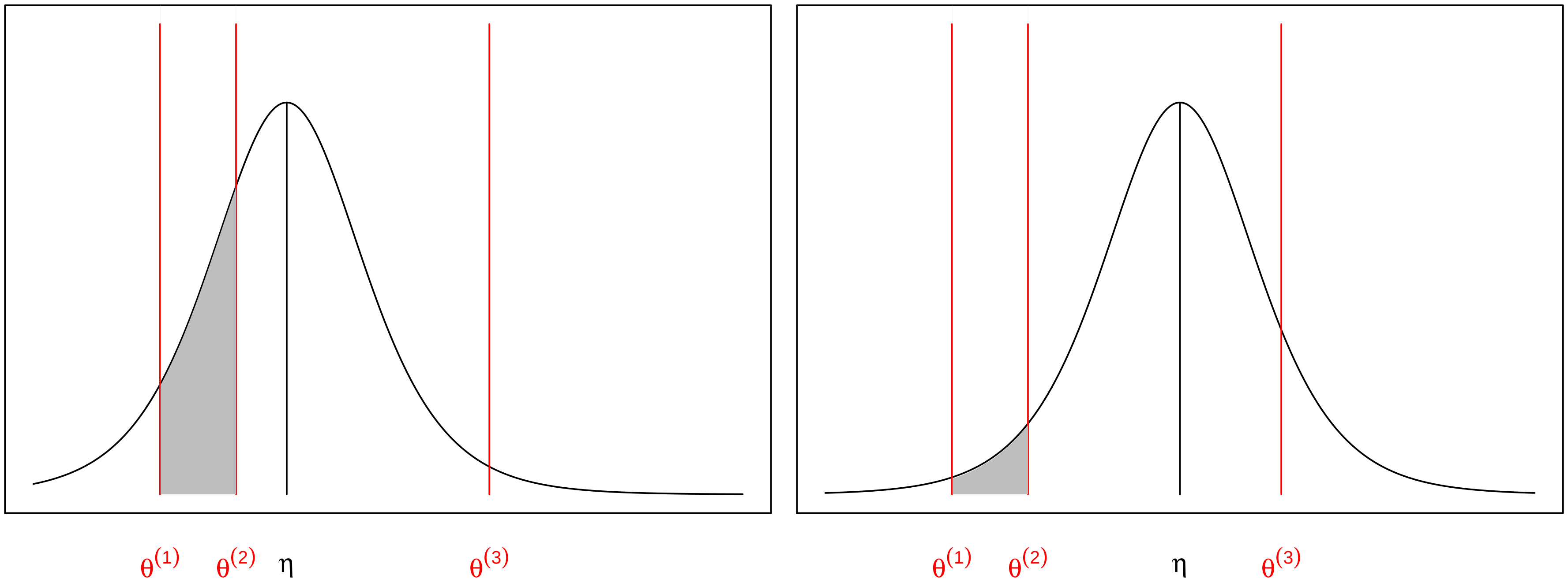
Empirical spatial effect.

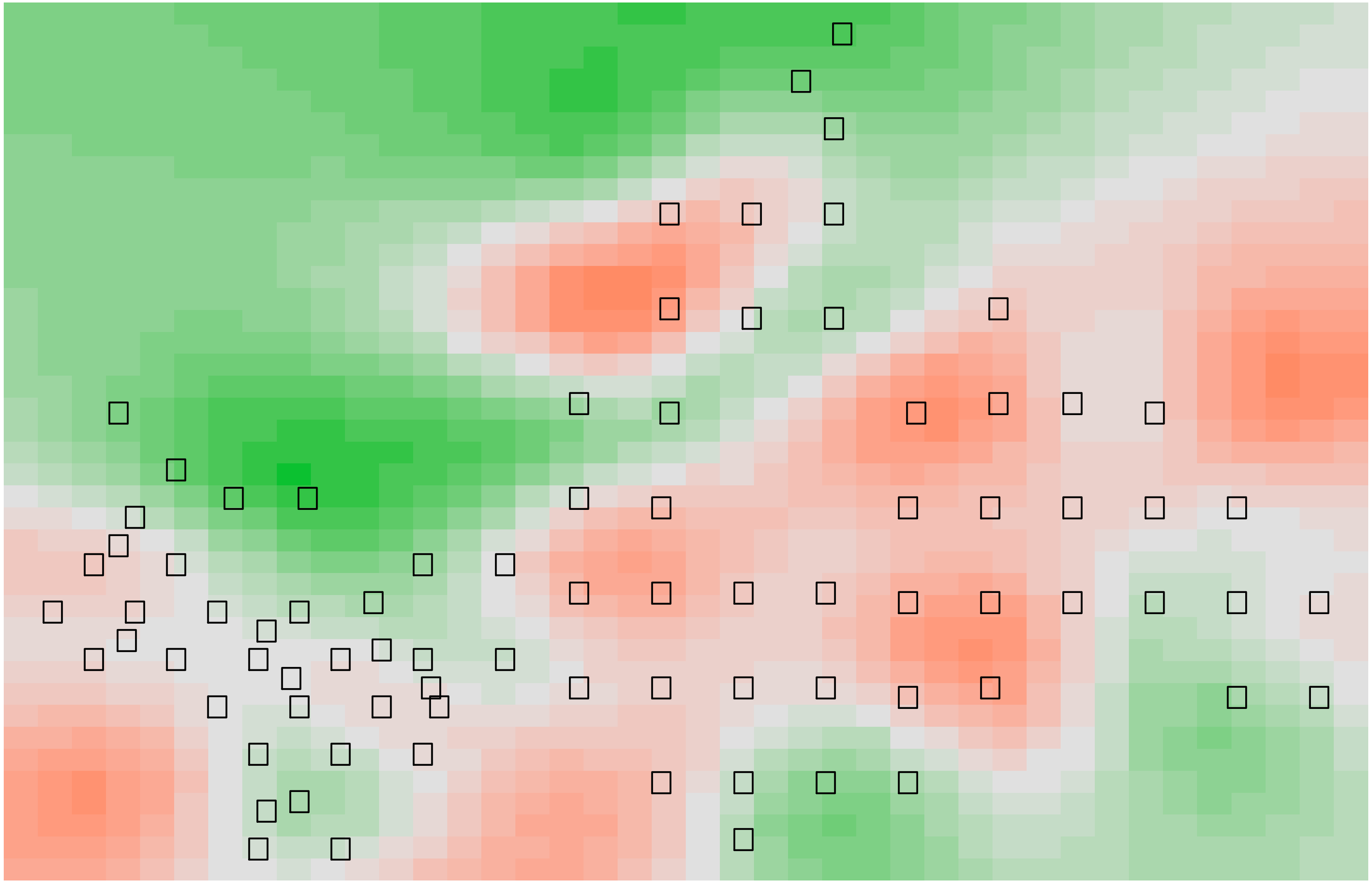
- Cumulative probit model:

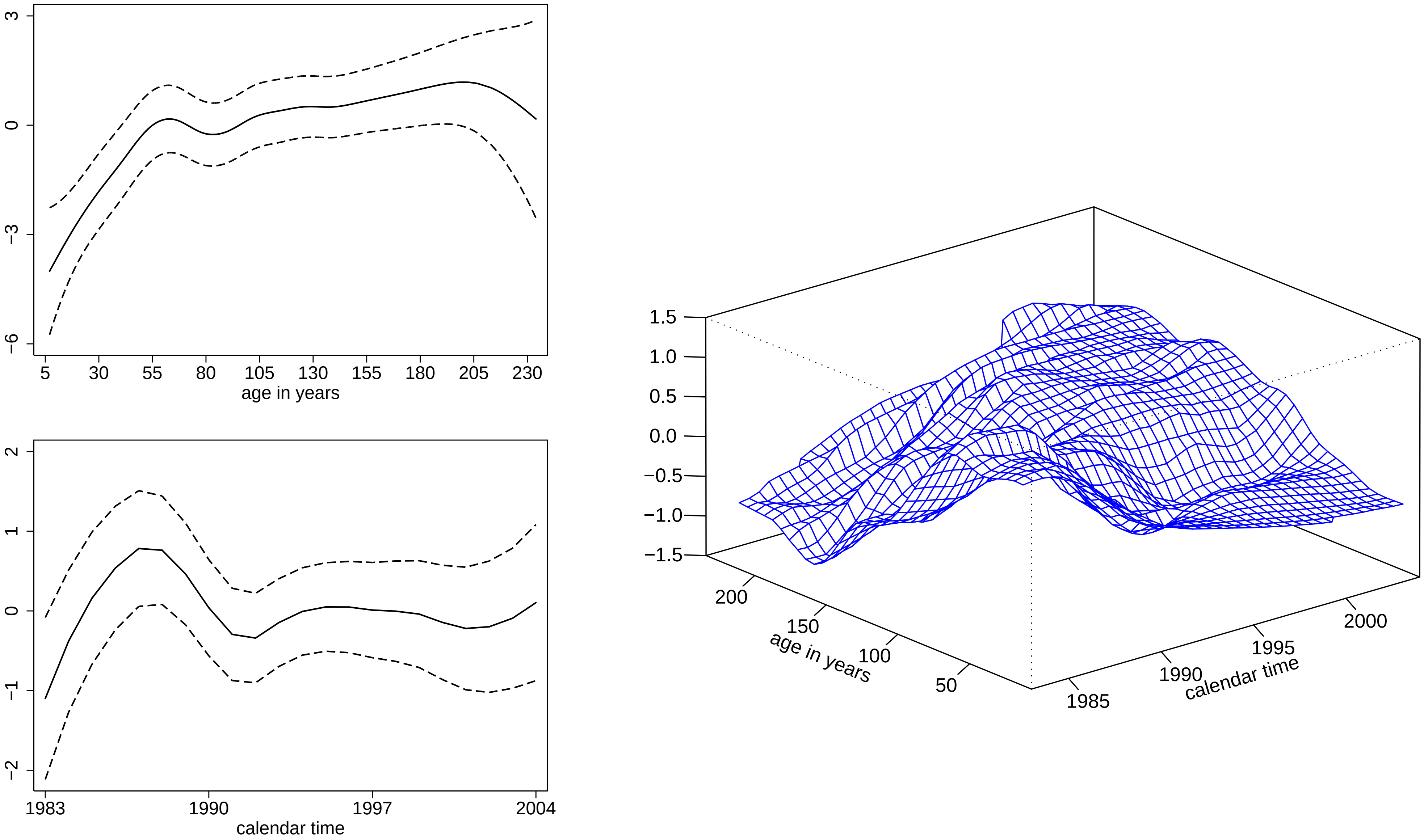
$$P(y_{it} \leq r) = \Phi\left(\theta^{(r)} - \eta_{it}\right)$$

with standard normal cdf Φ, thresholds $-\infty = \theta^{(0)} < \theta^{(1)} < \theta^{(2)} < \theta^{(3)} = \infty$ and

$$\eta_{it} = f_1(t) + f_2(age_{it}) + f_3(t, age_{it}) + f_{spat}(s_i) + u_{it}'\gamma$$






3
0
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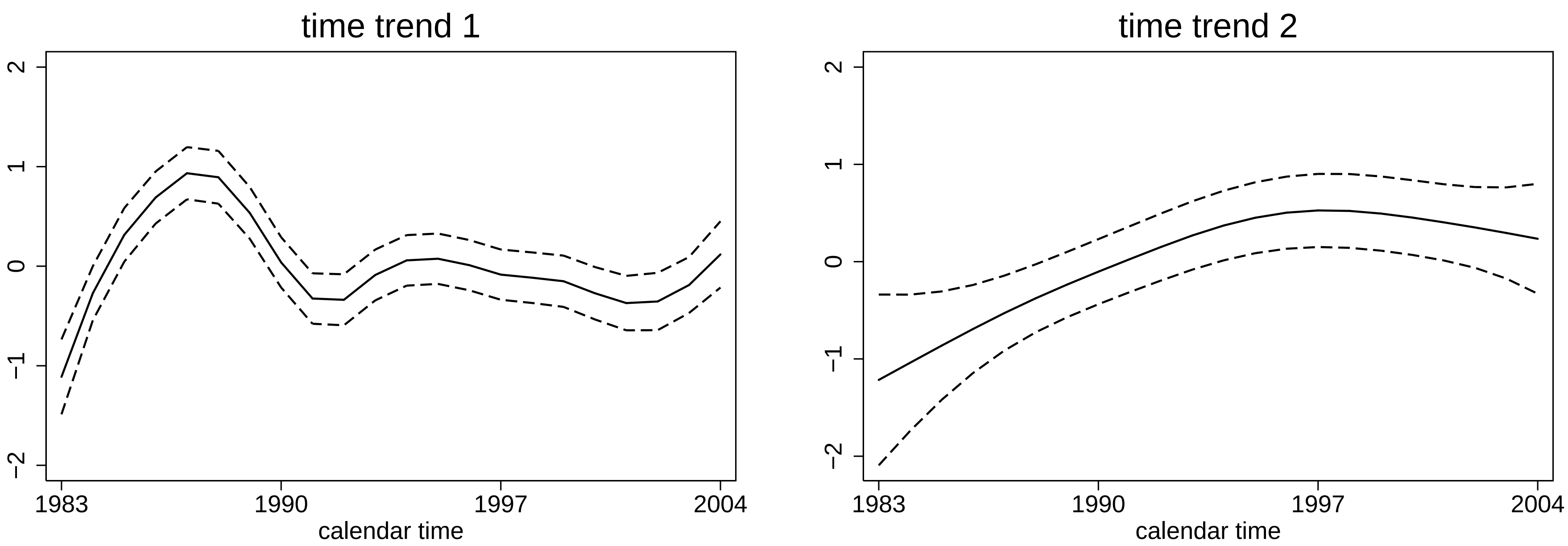
5
30
55
80
105
130
155
180
205
230
age in years
2
1
0
−1
−2
1983
1990
1997
2004
calendar time
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
200
150
100
50
age in years
1985
1990
1995
2000
calendar time

- Category-specific trends:

$$P(y_{it} \leq r) = \Phi \left[\theta^{(r)} - f_1^{(r)}(t) - f_2(age_{it}) - f_{spat}(s_i) - u_{it}'\gamma \right]$$

- More complicated constraints:

$$-\infty < \theta^{(1)} - f_1^{(1)}(t) < \theta^{(2)} - f_1^{(2)}(t) < \infty \qquad \text{for all } t.$$



Summary

- Flexible semiparametric regression models for geoadditive data structures.
- Fully automated Bayesian inferential procedures.
- Similar types of models are available for extended Cox-type hazard regression models:
  - Joint estimation of covariate effects and baseline hazard rate.
  - Time-varying effect to overcome proportional hazards.
  - Interval, left, and right censored survival times.
- A place called home:

`http://www.stat.uni-muenchen.de/~kneib`